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$\mathbb{Z}_N$ domain walls in hot $\mathcal{N}=4$ SYM at weak and strong coupling

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Abstract: We study the tensions of domain walls in the deconfined phase of $\mathcal{N}=4$ SUSY Yang-Mills theory on $\mathbb{R}^3 \times S^1$, at weak and strong coupling. We calculate the k-wall tension at one-loop order and find that it is proportional to $k(N-k)$ (Casimir scaling). The two-loops analysis suggests that Casimir scaling persists to this order. The strong coupling calculation is performed by using the AdS/CFT correspondence. We argue that the k-wall should be identified with an NS5-brane wrapping an S^4 inside S^5 in the AdS-Schwarzschild$\times S^5$ background in Type IIB string theory. The tension at strong coupling is compared with the weak coupling result. We also compare our results with those from lattice simulations in pure Yang-Mills theory.

Keywords: AdS-CFT Correspondence, Thermal Field Theory.

Contents

1. Introduction

In SU(N) gauge theories with adjoint matter the Polyakov-loop expectation value serves as an order parameter for the deconfinement transition [1, 2]. In the confining phase, the centre of the gauge group, $\mathbb{Z}_N$ for SU(N), is unbroken and the Polyakov loop expectation value is zero. In the deconfined phase on the other hand, the centre is spontaneously broken and the Polyakov loop expectation value is an element of the centre, $P = \exp(i2\pi k/N)$. When the discrete $\mathbb{Z}_N$ symmetry is spontaneously broken there exist domain walls that interpolate between the various vacua. The "fundamental" domain wall interpolates between two adjacent vacua. More generally, a domain wall which interpolates between one vacuum $P = \exp(i2\pi l/N)$, and another with $P = \exp i2\pi(l + k)/N$, is called a k-wall. By definition, it is obvious that all the properties of the k-wall (e.g. tension or width) should be invariant under $k \to k + N$. Charge conjugation implies invariance under $k \to N - k$ as well.

This paper is devoted to the study of domain walls in the deconfined phase of $\mathcal{N} = 4$ SUSY Yang-Mills theory on $\mathbb{R}^3 \times S^1$, with supersymmetry breaking boundary conditions for fermions. $\mathcal{N} = 4$ SYM is a useful toy model, since the domain walls can be analyzed at both weak and strong coupling. The weak coupling calculation is performed by a perturbative analysis, while the strong coupling analysis is performed by using the AdS/CFT correspondence.

Not much is known about deconfined phase domain walls of the $\mathcal{N} = 4$ theory. The literature that we will briefly review momentarily, is almost strictly devoted to domain walls in pure Yang-Mills theory. The tension of the domain walls in pure Yang-Mills theory was computed at high temperature and consequently, weak coupling, at one-loop order, in the seminal papers of Bhattacharya et al. [3, 4]. The result is

$$\mathcal{T}_k = k(N-k)\frac{4\pi^2}{3\sqrt{3}}\frac{T^2}{\sqrt{g_{\mathrm{YM}}^2(T)N}}\,. \tag{1.1}$$

where $g_{\mathrm{YM}}(T)$ is the running gauge coupling at temperature T. The above result (1.1) was extend to two-loop order in [5]. In particular, the tension up to two-loop order still exhibits a Casimir scaling, namely $\mathcal{T}_k \propto k(N-k)$. The three-loop analysis [6] reveals a deviation from Casimir scaling. The tension of domain walls was also measured by lattice simulations [7 − 9]. The conclusion of [7 − 9] is that Casimir scaling holds within the measurement error to low-temperatures (close to T_c), where perturbation theory is not valid anymore. Other aspects of deconfining phase domain walls and of particular significance for us, the relation between domain walls and spatial 't Hooft loops were investigated in [10].[1]

Within the AdS/CFT framework, domain walls in the deconfined phase of $\mathcal{N} = 4$ SYM, in the large-N limit and strong 't Hooft coupling, were discussed by Aharony and Witten [12]. They proposed that the fundamental domain wall should be identified with a D1-brane, reinforcing the connection between $\mathbb{Z}_N$ walls and spatial 't Hooft loops. Their work is reviewed in section 5.

The purpose of this paper is two-fold: (i) To carry out the perturbative analysis of [3, 5] for the $\mathcal{N} = 4$ SUSY Yang-Mills theory. (ii) To extended the analysis of [12] to the case of a k-wall. To this end, we propose that the k-wall is an NS5-brane (or a D5-brane in the S-dual theory) that wraps an S^4 inside the S^5 in the Type IIB string theory geometry, dual to the strongly coupled $\mathcal{N} = 4$ theory. The geometry dual to the Euclidean field theory on $\mathbb{R}^3 \times S^1$ is the Euclidean AdS-Schwarzschild black hole.

Our main results are as follows: at the one-loop order we find that the tension of the k-wall in the $\mathcal{N} = 4$ theory is

$$\mathcal{T}_k = k(N-k)\,\sqrt{2}\pi^3\,\frac{T^2}{\sqrt{\lambda}}\,; \qquad \lambda = g_{\mathrm{YM}}^2 N \ll 1 \tag{1.2}$$

[1] The physical status of these domain walls for the real time hot gauge theory in 3+1 dimensions has been questioned in [11] and other works. We will, however, only view the $\mathbb{Z}_N$ walls as interfaces in the effective 3 dimensional theory on $\mathbb{R}^3 \times S^1$ where the circle is interpreted as a spatial direction.

We extend our perturbative analysis beyond one-loop and argue that up to two loops the tension exhibits a Casimir scaling as in the pure YM case. Although we did not carry out the three-loop calculation, we do not expect that the Casimir scaling behaviour will hold beyond two loops.

In the limit of strong 't Hooft coupling $\lambda \gg 1$, in the large N limit, we find that the IIB supergravity dual yields a domain wall tension which is no longer proportional to $k(N-k)$. Instead we find that

$$\mathcal{T}_k = \tfrac{4}{3}\pi\, N^2\, \frac{T^2}{\sqrt{\lambda}}\, \sin^3\alpha_k\,, \qquad \frac{1}{2}\sin 2\alpha_k - \alpha_k = \pi\left(\frac{k}{N}-1\right). \tag{1.3}$$

This formula is valid when $N \to \infty$ while k/N is kept fixed. It is worth mentioning that (1.3) is extremely well approximated, within a maximum error of 2%, by a different expression [13]

$$\mathcal{T}_k \approx 2\pi\, N^2 \frac{T^2}{\sqrt{\lambda}} \left(\sin\pi\frac{k}{N} - \frac{1}{3}\left(\sin\pi\frac{k}{N}\right)^{\frac{3}{2}}\right). \tag{1.4}$$

The paper is organized as follows: section 2 is devoted to preliminary definitions and review. In section 3 we carry out a detailed one-loop calculation of the k-wall tension. In section 4 we argue that the Casimir scaling behavior should persist at two-loop. In section 5 we calculate the wall tension at strong 't Hooft coupling by using supergravity. In section 6 we discuss out results.

2. Preliminaries

The order parameter for the deconfinement transition in SU(N) gauge theories, with matter transforming in the adjoint representation, is the Polyakov loop. It is strictly only an order parameter in the Euclidean thermal theory and is defined as the Wilson line around the (compactified) Euclidean time direction,

$$P(x) = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\,\exp\left(i\,g_{\mathrm{YM}}\int_0^\beta A_0(x,t)dt\right),\ \beta = \frac{1}{T}. \tag{2.1}$$

At high temperatures, across the deconfinement transition the Polyakov loop acquires an expectation value, breaking the $\mathbb{Z}_N$ center symmetry and signalling a deconfined phase [1, 2]. The spontaneous breaking of the $\mathbb{Z}_N$ symmetry leads to N distinct phases labelled by the N roots of unity. When the Euclidean gauge theory on $\mathbb{R}^3 \times S^1$ is viewed as a three dimensional effective field theory, one can imagine domain walls interpolating between regions with different expectation values of P. As first established in the works of [3, 4, 14], the explicit profiles of the $\mathbb{Z}_N$ interfaces and their tensions can be computed at weak coupling, utilizing high temperature perturbative and semiclassical methods. The $\mathbb{Z}_N$ domain wall is like an instanton in a one dimensional effective theory describing the profile perpendicular to the interface. The perturbative description works because the thickness of the domain wall is set by $(g_{\mathrm{YM}}N^{\frac{1}{2}}T)^{-1}$, the Debye screening scale.

In what follows, we will review (cf. [14]) the semiclassical description of the $\mathbb{Z}_N$ interfaces and apply it to the $\mathcal{N}=4$ theory at high temperature and weak gauge coupling. The first step in this computation involves parametrizing the varying expectation value of the Polyakov loop across the domain wall, by a classical background value for A_0. Thus the temporal gauge field A_0 can be split into classical and quantum parts,

$$A_0 = A_0^{\rm cl} + A_0^{\rm qu}. \tag{2.2}$$

2.1 The fundamental wall or $k = 1$

In the three dimensional effective theory, the domain interface can be thought of as a string-like object moving through time. Taking this interface to span the $x-y$ plane, the different vacua then sit at different values of z. Focusing on the classical, or background field part, we pick an ansatz where $A_0^{\rm cl}$ is expressed in terms of the diagonal traceless generator t_N,

$$A_0^{\rm cl} = \frac{2\pi T}{g_{\rm YM} N} \frac{t_N}{B(N)}\, q(z)\,; \quad t_N = B(N)\ {\rm Diag}[\underbrace{1,1,1,\ldots,1}_{N-1\,\text{entries}}, 1-N] \tag{2.3a}$$

$$B(N) = \frac{1}{\sqrt{2\,N\,(N-1)}}. \tag{2.3b}$$

The spatial components of the gauge field are chosen to vanish, $A_i = 0$ for $i = 1,2,3$. Importantly, the profile function q parametrizes the N different vacua. This can be seen more explicitly by evaluating the Polyakov loop order parameter with the above ansatz on a constant background field $A_0^{\rm cl}$ (the explicit z dependence will be dealt with later),

$$P = \frac{1}{N}\left[(N-1)e^{\frac{2\pi i q}{N}} + e^{\frac{2\pi i q(1-N)}{N}}\right]. \tag{2.4}$$

Allowing q to take on one of N integer values, $q = 0,1,2,\ldots,N-1$, we can scan through each of the N vacua labelled by $P = e^{\frac{2\pi i q}{N}}$. As the vacua are labelled by integer values of q, the domain interface itself will be characterized by non-integer values of the profile function $q(z)$ interpolating between two vacua.

The ansatz (2.3a) can be used to describe a wall that interpolates between two $\mathbb{Z}_N$ phases labelled by consecutive integers. Since all the vacua are physically equivalent, to discuss the wall tension, we may focus on the interface between $q = 0$ and $q = 1$ without loss of generality. Therefore, we may think of $q(z)$ as interpolating between $q(z = 0) = 0$ and $q(z = L) = 1$, where L is the extent or thickness of the interface between neighbouring vacua.

Up to this point, the walls under scrutiny have been the fundamental or $k = 1$ walls. To consider walls between vacua with multiple $\mathbb{Z}_N$ "charge" difference, k, the ansatz for $A_0^{\rm cl}$ must be modified as below.

2.2 The k-wall ansatz

We first set out our conventions for the N^2-1 generators of SU(N). Separating them into the $N-1$ diagonal generators of the Cartan subalgebra and $N(N-1)$ off-diagonal or ladder generators, the Cartan elements are of the form,

$$t_{\rm diag} \equiv t_i = B(i)\ {\rm Diag}[\underbrace{1,1,1,\ldots,1}_{i-1\,\text{entries}}, 1-i, \underbrace{0,0,\ldots,0}_{N-i\,\text{entries}}] \qquad i \in [2,N] \tag{2.5}$$

The normalization $B(i)$ as defined in (2.3b) ensures that

$$\mathrm{Tr}(t_i\, t_j) = \frac{1}{2}\delta_{ij}. \tag{2.6}$$

For every Cartan generator t_i, we can define $2(i-1)$ ladder generators, t_{ij} with one non-zero element:

$$t_{ij}^{mn} = \frac{1}{\sqrt{2}}\delta_i^n \delta_j^m\,, \qquad j \in [1, i-1]. \tag{2.7}$$

t_{ij} provides the off-diagonal generators with the non-zero matrix elements in the upper right half, while the lower left off-diagonal generators are given by the transpose, t_{ji}. The off-diagonal ladder generators are normalized in the following way

$$\mathrm{Tr}(t_{ij}\, t_{j'i'}) = \frac{1}{2}\,\delta_{ii'}\,\delta_{jj'}. \tag{2.8}$$

The algebra of the generators simplifies significantly in this basis, with the only non-vanishing commutators being

$$[t_i, t_{ij}] = NB(i)\, t_{ij}\,; \quad [t_i, t_{ji}] = -NB(i)\, t_{ji}. \tag{2.9}$$

Returning to the idea of an interface between two vacua labelled by generic integers, a so called k-wall, a modified ansatz is required. As a k-wall is an interface between two vacua with a $\mathbb{Z}_N$ charge difference k, the ansatz for $A_0^{\rm cl}$ is chosen to be proportional to the hypercharge matrix Y_k

$$A_0^{\rm cl} = \frac{2\pi T}{g_{\rm YM} N}\, q(z)\, Y_k. \tag{2.10}$$

where Y_k is defined as

$$Y_k \equiv \mathrm{Diag}[\underbrace{k, k, k, \ldots, k}_{N-k\,\text{entries}}, \underbrace{k-N, k-N, \ldots, k-N}_{k\,\text{entries}}]\,, \qquad k \in [1, N] \tag{2.11}$$

As with t_N, Y_k is traceless, and the resulting ansatz will be symmetric under $k \leftrightarrow N-k$ which is required by $\mathbb{Z}_N$ invariance of the theory.

Applying these modifications to the order parameter, the role of q is now clear. Previously, for $k=1$, the parameter q defined each vacuum individually when integer valued, and non-integer values of q characterized a point within an interface. For the k-walls with $k > 1$, it is no longer q, but the product kq that specifies a given vacuum for integer values of q; q now becomes a parameter varying across the k-wall, as before from $q(0) = 0$ to $q(L) = 1$. The Polyakov loop order parameter, for this ansatz is

$$P = \frac{1}{N}\left[(N-k) e^{\frac{2\pi i}{N} k\, q} + k\, e^{\frac{2\pi i}{N}(k-N)\, q}\right] \tag{2.12}$$

with $P = 1$ at $q = 0$ as before, and $P = e^{\frac{2\pi i}{N} k}$ when $q = 1$.

3. k-Wall Tension in $\mathcal{N}=4$ SYM

Having specified the ansatz for the k-wall solution, our aim now is to determine the tension of the k-wall in $\mathcal{N}=4$ SYM. The basic idea is to insert the classical or background profile for the $\mathbb{Z}_N$ instanton and use weak coupling to expand in the quantum fluctuations around the background configuration. The quantum fluctuations induce a one-loop effective potential for $q(z)$, which is then used to determine the wall solution and its tension. Crucially, it is necessary to ensure self-consistently that the resulting configuration can be understood at weak coupling. We will elaborate on this subsequently.

At the classical level, as shall be shown, there is no interface solution, so one-loop effects must be included. The only terms in the action of $\mathcal{N}=4$ SYM that we need to focus attention on, are those that involve the interactions of the background gauge configuration $A_0^{\rm cl}$ with quantum fluctuations:

$$S=\int d^3x\int_0^\beta d\tau\left\{\mathrm{Tr}\left[\frac{1}{2}(F_{\mu\nu})^2\right]+\mathrm{Tr}\left[\sum_{A=1}^{4}\overline{\psi}_A \not{D}\,\psi_A\right]+\mathrm{Tr}\left[\sum_{i=1}^{6}\frac{1}{2}D_\mu\phi_i D^\mu\phi_i\right]+\ldots\right\} \tag{3.1}$$

The relevant portion of the action includes only kinetic terms for the four Majorana fermions and six real scalars, and their interactions with the background field through the gauge covariant derivative. We have omitted in the above, the Yukawa couplings and the $\mathcal{N}=4$ quartic scalar potential. Ultimately, integrating over the fluctuating quantum fields ϕ_i, ψ_A, the gauge fluctuations $A_\mu^{\rm qu}$, and the ghosts arising from gauge fixing, will generate an effective potential for the classical gauge fields.

Working in Euclidean space, on $\mathbb{R}^3\times S^1$ with antiperiodic boundary conditions for fermions around the thermal circle, we will treat each of the quantum fluctuations separately below.

3.1 Gauge field fluctuations

The one-loop calculations outlined in this section follow essentially standard steps, however we include them here for completeness. As previously seen, the gauge field A^μ consists of classical and quantum parts, thus the gauge part of the action can be separated accordingly

$$S_A=S_A^{\rm cl}+S_A^{\rm qu}. \tag{3.2}$$

Letting q be a general function of z, and using the fact that the only non-zero classical gauge field is $A_0^{\rm cl}$, the classical action can be evaluated simply on this background,

$$\mathcal{L}_A^{\rm cl}\;=\;\mathrm{Tr}\left[\frac{1}{2}(F_{\mu\nu})^2\right]\;=\;\mathrm{Tr}\left[(\partial_z A_0^{\rm cl})^2\right]=\mathrm{Tr}\left[\frac{4\pi^2T^2}{g_{\rm YM}^2N^2}(\partial_z q)^2Y_k^2\right]. \tag{3.3}$$

Using $\mathrm{Tr}Y_k^2=Nk(N-k)$, we obtain

$$S_A^{\rm cl}=\frac{4\pi^2T^2}{g_{\rm YM}^2N}\,k(N-k)\int d^3x\int_0^\beta d\tau\,(\partial_z q)^2. \tag{3.4}$$

As mentioned briefly at the start of section 3, the classical action alone is not enough to show the existence of k-walls, since it is only sensitive to the gradient energy that is minimized by a constant q solution. To find a k-wall solution, the action needs to be calculated beyond tree level, at one-loop order, self-consistently at weak coupling.

To treat the gauge field fluctuations at one-loop or quadratic order, we shift attention to the quantum part of the action wherein we must fix a gauge. We employ the usual background field R_ξ gauges to obtain the action for the quantum fluctuations of the gauge field

$$\mathcal{L}_A^{\rm qu} = {\rm Tr}\left[\frac{1}{2}(F_{\mu\nu}^{\rm qu})^2\right] + {\rm Tr}\left[\frac{1}{\xi}\left(D_{\rm cl}^\mu A_\mu^{\rm qu}\right)^2\right] + {\rm Tr}\left[\bar\eta\left(-D_{\rm cl}^2\right)\eta\right] \tag{3.5}$$

with $\bar\eta$ and η being the Fadeev-Popov ghosts, and the adjoint covariant derivative D_μ and $D_\mu^{\rm cl}$ defined thus,

$$D_\mu = \partial_\mu - ig_{\rm YM}[A_\mu,\, .], \quad D_\mu^{\rm cl} = \partial_\mu - ig_{\rm YM}[A_\mu^{\rm cl},\, .]\,. \tag{3.6}$$

At the one-loop order we can completely ignore the interactions between different quantum fluctuations. This amounts to replacing the full covariant derivative D_μ with $D_{\mu{\rm cl}}$ which is gauge-covariant with respect to the background. Integrating by parts and assuming that the background field is *constant*, the action for the quantum fluctuations becomes

$$S_A^{\rm qu} = \int d^3x \int_0^\beta d\tau\, {\rm Tr}\left[A_\mu^{\rm qu}\left(-D_{\rm cl}^2\, g^{\mu\nu} + (1-\frac{1}{\xi})D_{\rm cl}^\mu D_{\rm cl}^\nu\right)A_\nu^{\rm qu}\right] + {\rm Tr}\left[\bar\eta\left(-D_{\rm cl}^\mu D_{\rm cl}^\mu\right)\eta\right]. \tag{3.7}$$

Technically, it is important to note that we are assuming a constant background field and therefore we will obtain an effective potential for constant field configurations only. Nevertheless we will employ the same effective potential to look for non-constant domain wall profiles. Hence this really requires the profile function to be appropriately slowly varying. Performing the functional integral over the gauge fluctuations $A_\mu^{\rm qu}$, and the ghost fields, the one-loop contribution is

$$S_A^{\rm qu} = \frac{1}{2}{\rm Tr}\left[\ln\left(-D_{\rm cl}^2\, g^{\mu\nu} + (1-\frac{1}{\xi})D_{\rm cl}^\mu D_{\rm cl}^\nu\right)\right] - {\rm Tr}\left[\ln\left(-D_{\rm cl}^2\right)\right]. \tag{3.8}$$

The effective action can be shown to be independent of the gauge fixing parameter ξ due to the commutativity of the covariant derivatives for constant backgrounds. Thus, at least for slowly varying interface profiles we are guaranteed to obtain gauge-invariant results and, in particular, we will choose the Feynman gauge, $\xi = 1$ so that,

$$S_A^{\rm qu} = -\frac{1}{2}{\rm Tr}\ln(-D_{\rm cl}^2). \tag{3.9}$$

As the background field $A_0^{\rm cl}(z)$ present in the adjoint covariant derivative is only non-zero along the compact direction, τ, it reduces to an ordinary derivative in the transverse directions, x, y and z. In the compact direction the background field is proportional to the matrix Y_k, and being diagonal with N elements, there exist non-trivial contributions to the covariant derivative when acting upon ladder generators t_{ij} and t_{ji} (see for example eq. (2.9)). Following the notation of [14]

$$D_0^{\rm cl}\, t_{ij} = (\partial_0 - 2\pi i T q)\, t_{ij} \equiv D_0^+\, t_{ij}\,, \tag{3.10}$$

$$D_0^{\rm cl}\, t_{ji} = (\partial_0 + 2\pi i T q)\, t_{ji} \equiv D_0^-\, t_{ji}\,. \tag{3.11}$$

The commutator of Y_k with the ladder operators for $\mathrm{SU}(N)$, has very similar properties to the commutator in eq. (2.9) while there are significant differences:

$$[Y_k, t_{ij}] = N t_{ij} \qquad [Y_k, t_{ji}] = -N t_{ji}\,, \qquad i \in [N-k, N]\,, \qquad j \in [1, N-k] \tag{3.12}$$

All other commutators vanish. The full non-trivial q dependence comes from the action of the covariant derivatives on the ladder generators; equivalently, from integrating out all off-diagonal fluctuations that do not commute with Y_k. For this reason we may replace the covariant derivatives with

$$D_{\rm cl} \to (D_0^\pm, \vec{\partial}\,). \tag{3.13}$$

Fourier transforming to Euclidean momentum space, the temporal derivative ∂_0 may be replaced by the Matsubara frequencies p_0,

$$i\partial_0 \to p_0 = 2\pi n T\,, \qquad n \in \mathbb{Z}. \tag{3.14}$$

The action of the covariant derivatives on ladder operator-like fluctuations is, from eq. (3.11),

$$iD_0^\pm \to p_0^\pm = 2\pi T(n \pm q). \tag{3.15}$$

Since there are precisely $k(N-k)$ fluctuations (3.12) which yield a non-zero contribution to the effective action at one-loop and the sum over the Matsubara modes includes both positive and negative integers, we have

$$S_A^{\rm qu} \,=\, 2k(N-k)\, V_{\rm tr} L \sum_{n=-\infty}^{+\infty} \int T \frac{d^3\mathbf{p}}{(2\pi)^3} \ln\left((p_0^+)^2 + \mathbf{p}^2\right). \tag{3.16}$$

Here $V_{\rm tr}$ is the volume of the space transverse to the z-axis

$$V_{\rm tr} = L_1 L_2 \beta\,, \tag{3.17}$$

with L_1, L_2 being the extents of the system in the x and y directions respectively. As the Euclidean time circle is compactified, the k-wall can be viewed as being smeared along this direction. Now it remains to determine the dependence of the one-loop effective action on q. Up to irrelevant additive constants, the q-dependence is determined by the variation of $S_A^{\rm qu}$ with respect to q,

$$\begin{aligned}\frac{1}{2\pi T}\frac{\partial S_A^{\rm qu}}{\partial q} \,&=\, 4k(N-k) V_{\rm tr}\, L \sum_{n=-\infty}^{+\infty} \int T \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{p_0^+}{(p_0^+)^2 + \mathbf{p}^2}\right) \\ &=\, -4k(N-k) V_{\rm tr}\, L\, \pi T^3 \sum_{n=-\infty}^{+\infty} (n+q)|n+q|,\end{aligned} \tag{3.18}$$

where the final result follows from standard expressions for the regulated integral using dimensional regularisation. Using zeta function regularisation, the explicitly divergent sum over n can be controlled quite elegantly,[2]

$$\sum_{n=-\infty}^{+\infty}(n+q)|n+q| = \sum_{n=0}^{+\infty}\left[(n+q)^2-(n+1-q)^2\right] = \zeta(-2,q)-\zeta(-2,1-q). \qquad (3.19)$$

Remarkably, this particular form of the Hurwitz zeta function is a simple polynomial in q

$$\zeta(-2,q) = -\frac{1}{12}\frac{d}{dq}\left[q^2(1-q)^2\right]. \qquad (3.20)$$

Finally we obtain the one-loop effective potential for slowly varying $q(z)$, (up to an additive constant)

$$S_A^{\rm qu} = \frac{4}{3}k(N-k)V_{\rm tr}\,\pi^2T^4\int_0^L dz\, q^2(1-q)^2. \qquad (3.21)$$

The potential is manifestly invariant under $q\to 1-q$ and has minima at $q=0$ and $q=1$. We can therefore have a 'kink'-like configuration interpolating between these two vacua. Combining the quantum one-loop action along with the classical kinetic term calculated earlier, the total effective action can be compactly expressed, after a coordinate rescaling $z\to z'=\sqrt{g_{\rm YM}^2N/3}\,Tz$ as,

$$S_A = \frac{4\pi^2T^3}{\sqrt{3N}g_{\rm YM}}\,k(N-k)\,V_{\rm tr}\int_0^{L'}dz'\left[\left(\frac{\partial q}{\partial z'}\right)^2+q^2(1-q)^2\right]. \qquad (3.22)$$

The double well potential, represents the so-called "q-valley". Due to the rescaling of $z\to z'$, the upper limit of integration L is also effectively rescaled: $L\to L'=\sqrt{g_{\rm YM}^2N/3}\,TL$. The large volume limit corresponds to $\sqrt{g_{\rm YM}^2N}TL\to\infty$ which can also be viewed as the three dimensional limit when the thermal circle shrinks to zero size.

We can now self-consistently justify the use of the constant-q effective potential to infer the existence of the spatially varying domain wall. In terms of the z' coordinate, it is clear that the width of the domain wall is a number $\sim\mathcal{O}(1)$. In physical units, the width of the domain wall is then set by $(\sqrt{g_{\rm YM}^2N}T)^{-1}$, which is the Debye or electric screening length. At weak gauge coupling (or weak 't Hooft coupling at large N), this is much larger than the typical thermal wavelength T^{-1}, of the perturbative degrees of freedom. Thus the domain wall is thick and a slowly varying configuration. Furthermore, since the scale of variation is set by the Debye scale, the wall and its properties should be accessible in perturbation theory. If the wall thickness had been set by the magnetic scale $(g_{\rm YM}^2NT)^{-1}$, the non-perturbative scale of the three dimensional effective theory, the perturbative description above would be invalidated.

Having reviewed the perturbative gauge field contributions to the physics of the domain walls, let us now turn to the matter fields in the adjoint representation in the $\mathcal{N}=4$ theory.

[2]The Hurwitz zeta function is defined as $\zeta(l,m)=\sum_{n=0}^{+\infty}(n+m)^{-l}$.

3.2 Scalar field fluctuations

The $\mathcal{N}=4$ theory has six hermitian scalars transforming in the adjoint representation of the gauge group. We consider them first, due to the similarity in the calculation to the gauge field contribution, before turning to the fermion fields in section 3.3. The scalar part of the action coupled to the classical background gauge field is,

$$S_S = \int d^3x \int_0^\beta d\tau \,\mathrm{Tr}\left[\sum_{i=1}^{n_s} \frac{1}{2}\left(D_\mu^{\rm cl}\phi_i D^\mu_{\rm cl}\phi_i\right)\right] \tag{3.23}$$

where $n_s = 6$ is the number of real adjoint scalars, and we have ignored interactions of the fluctuations. Integrating out the scalar field fluctuations we have

$$S_S = \frac{n_s}{2}\int d^3x \int_0^\beta d\tau \,\mathrm{Tr}\ln(-D_{\rm cl}^2). \tag{3.24}$$

This is exactly the same as eq. (3.9) up to the overall normalization and generates the one-loop potential

$$S_S = \frac{4\pi^2 T^3}{\sqrt{3N g_{\rm YM}^2}} k(N-k)\frac{n_s}{2} V_{\rm tr}\int_0^{L'} dz'\, q^2(1-q)^2. \tag{3.25}$$

3.3 Fermionic contributions at one loop

Finally there are the $n_f = 4$ fermions transforming as a **4** of the SO(6) R-symmetry. These play a crucial role at finite temperature since they have antiperiodic, supersymmetry breaking boundary conditions around the thermal circle. At quadratic order in the fluctuations

$$S_F = \int d^3x \int_0^\beta d\tau \,\mathrm{Tr}\left[\sum_{A=1}^{n_f} \overline{\psi}_A \not{D}\, \psi_A\right]. \tag{3.26}$$

Working in Euclidean space, the Dirac gamma matrices are

$$\gamma^{1,2,3} = \begin{pmatrix} 0 & -i\sigma_{1,2,3} \\ i\sigma_{1,2,3} & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \tag{3.27}$$

where σ^i are the standard Pauli matrices. The functional integral over the fermion fields then yields the Pfaffian of the Dirac operator, since ψ_A and $\bar{\psi}_A$ are not independent due to the Majorana condition,

$$S_F = -\int d^3x \int_0^\beta d\tau \, n_f \,\mathrm{Tr}\ln\left[-((D_0^{\rm cl})^2 + \nabla^2)\right]. \tag{3.28}$$

Despite the formal similarity to the bosonic contributions, it is at this point that the analysis departs from that for scalar and gauge fluctuations.

Consider first the case (zero temperature) wherein the compact direction has *periodic* (SUSY preserving) boundary conditions for the fermions. The fluctuation determinant (3.28) would be identical to that of the bosons (and the opposite sign) to produce a one-loop action of the form $S_F = -n_f S_A^{\rm qu}$. With supersymmetric boundary conditions the three fluctuation terms at the one-loop level would cancel, leaving only the classical action,

$$S_{\rm Total} = S_A^{\rm cl} + S_A^{\rm qu} + S_S + S_F = S_A^{\rm cl} + (1 + n_s/2 - n_f) S_A^{\rm qu} = S_A^{\rm cl}. \tag{3.29}$$

The cancellation between bosons and fermions will persist at all loop orders for SUSY-preserving boundary conditions.

However, in the Euclidean thermal theory, the fermions have anti-periodic boundary conditions around the Euclidean time circle. The Matsubara frequencies are thus shifted to half-integer values $n \to n + \frac{1}{2}$, $n \in \mathbb{Z}$. The eigenvalues of the covariant derivative, eq. (3.15), acting on fermions is modified

$$iD_0^{\pm} \to p_0^{\pm} = 2\pi T \left(n + \frac{1}{2} \pm q \right). \tag{3.30}$$

This shift has a non-trivial effect on the one-loop effective potential and the fermi-bose cancellations will be absent. With a half-integer moding of the fermionic Matsubara frequencies, the sums over n for p_0^+ and p_0^- are no longer equivalent, and each sum must be evaluated separately,

$$S_F = -k(N-k) V_{\rm tr} LT\, n_f \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\ln\left[(p_0^+)^2 + \mathbf{p}^2 \right] + \ln\left[(p_0^-)^2 + \mathbf{p}^2 \right] \right). \tag{3.31}$$

Once again as before it is useful to take the variation of the action with q, in order to determine the q-dependence of the effective potential,

$$\frac{\partial S_F}{\partial q} = -2k(N-k)(2\pi T) V_{\rm tr} LT\, n_f \sum_{n=-\infty}^{+\infty} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{p_0^+}{(p_0^+)^2 + \mathbf{p}^2} - \frac{p_0^-}{(p_0^-)^2 + \mathbf{p}^2} \right). \tag{3.32}$$

Again, integrating over the spatial momenta $\mathbf{p}$ employing dimensional regularization,

$$\begin{aligned} \frac{\partial S_F}{\partial q} =& 2k(N-k)(2\pi T) V_{\rm tr} LT\, n_f\, \pi T^2 \\ & \times \sum_{n=-\infty}^{+\infty} \left[(n + 1/2 + q)\, |n + 1/2 + q| - (n + 1/2 - q)\, |n + 1/2 - q| \right]. \end{aligned} \tag{3.33}$$

Now, to regulate this sum with zeta functions, we consider each term in the sum over n individually and using the definition of the Hurwitz zeta function, the sums simplify to

$$\sum_{n=0}^{+\infty} (n + 1/2 + q)^2 - (n + 1/2 - q)^2 = \zeta\Big(-2, \frac{1}{2} + q \Big) - \zeta\Big(-2, \frac{1}{2} - q \Big) \tag{3.34}$$

$$\sum_{n'=0}^{+\infty} \left(n' - 1/2 + q \right)^2 - \left(n' + 3/2 - q \right)^2 = \zeta\Big(-2, -\frac{1}{2} + q \Big) - \zeta\Big(-2, \frac{3}{2} - q \Big). \tag{3.35}$$

The definition of the Hurwitz zeta function as a derivative, eq. (3.20) allows the action to be explicitly determined, up to integration constants,

$$\begin{aligned} S_F =& -\frac{4\pi^2 T^3}{\sqrt{3N} g_{\rm YM}} k(N-k) V_{\rm tr} \frac{n_f}{2} \\ & \times \int dz \left[\left(\frac{1}{2} + q \right)^2 \left(\frac{1}{2} - q \right)^2 + \left(-\frac{1}{2} + q \right)^2 \left(\frac{3}{2} - q \right)^2 \right]. \end{aligned} \tag{3.36}$$

It is obvious now that this fermionic action will not cancel against the quantum gauge and scalar parts. The effective potential is invariant under $q \to 1-q$.

3.4 Full one-loop effective action

Putting all the above ingredients together we obtain the full one-loop effective action for the interface

$$S_{\rm Total} = S_A + S_S + S_F \tag{3.37}$$

$$= \frac{4\pi^2 T^3}{\sqrt{3N} g_{\rm YM}^2}\, k(N-k)\, V_{\rm tr} \int dz' \left[\left(\frac{\partial q}{\partial z'}\right)^2 + (1+\frac{n_s}{2}) q^2 (1-q)^2 \right.$$
$$\left. - \frac{n_f}{2} \left\{ \left(\frac{1}{2}+q\right)^2 \left(\frac{1}{2}-q\right)^2 + \left(-\frac{1}{2}+q\right)^2 \left(\frac{3}{2}-q\right)^2 \right\} \right]. \tag{3.38}$$

Letting n_s and n_f take their explicit values in $\mathcal{N}=4$ SYM, the quantum effective action simplifies to

$$S_{\rm Total} = \frac{4\pi^2 T^3}{\sqrt{3 g_{\rm YM}^2 N}} k(N-k) V_{\rm tr} \int dz' \left[\left(\frac{\partial q}{\partial z'}\right)^2 + 6q(1-q) - 5/4 \right]. \tag{3.39}$$

It is a simple exercise to obtain the minimum action configuration that interpolates between the two vacua $q=0$ and $q=1$, satisfying $(dq/dz')^2 = 6q(1-q)$, so that the action for the kink or domain wall is

$$S_{\rm Total} = \frac{4\pi^2 T^3}{\sqrt{3 g_{\rm YM}^2 N}} k(N-k) V_{\rm tr}\, 2 \int_0^1 dq \sqrt{6q(1-q)} = \sqrt{2} \frac{\pi^3 T^3}{\sqrt{g_{\rm YM}^2 N}}\, V_{\rm tr}\, k(N-k). \tag{3.40}$$

We therefore conclude that the tension of the k-wall in the Euclidean high temperature, $\mathcal{N}=4$ theory at weak coupling is

$$\mathcal{T}_k = \sqrt{2}\, \pi^3 \frac{T^2}{\sqrt{g_{\rm YM}^2 N}}\, k(N-k). \tag{3.41}$$

where one factor of T has cancelled against the size of thermal circle in $V_{\rm tr}$, leaving us with the tension of a $1+1$ dimensional interface in three dimensions. We note firstly that the parametric dependence on the gauge coupling is the same as in ordinary Yang-Mills theory [3, 4, 14]. One difference is that unlike in pure Yang-Mills theory, the gauge coupling itself does not run and therefore does not depend on temperature. We are, however, free to choose an arbitrary weak coupling in $\mathcal{N}=4$ theory, $g_{\rm YM} \ll 1$. All other qualitative aspects of the solution are similar to pure Yang-Mills theory. Specifically, the wall is "fat" with a width set by the Debye screening length $(\sqrt{g_{\rm YM}^2 N} T)^{-1}$. Interestingly, for $k \sim \mathcal{O}(N^0)$, in the 't Hooft large-N limit, the k-wall tension scales as N^1 rather than N^2.

Finally, the one-loop calculation demonstrates a Casimir scaling law for the tension of the k-wall. It is not *a priori* clear that Casimir scaling will persist at higher loop orders, since at one-loop its origin is essentially kinematic. Next we will investigate whether Casimir scaling remains at the two-loop level in $\mathcal{N}=4$ theory.

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4. $\mathcal{N}=4$ at 2-loop

It has been shown [5, 6] in pure Yang-Mills theory that Casimir scaling of $\mathbb{Z}_N$ domain walls remains at two-loops, but is lost at three-loops. Below we adapt the arguments of the two-loop result for pure Yang-Mills theory to argue that the scaling for $\mathcal{N}=4$ theory will also be Casimir like at the two-loop order.

4.1 Pure YM at 2-loop

Consider first the 2-loop calculation of the domain wall tension in pure Yang-Mills theory in the deconfined phase. Let us first lay out our notation and conventions. Defining the structure constants of the SU(N) algebra and their normalisation as usual

$$i f^{a,b,c} = 2\,\mathrm{Tr}\left(\left[t^a, t^b\right] t^c\right), \qquad \left(f^{a,b,c}\right)^2 = \frac{1}{2}, \tag{4.1}$$

the indices a, b, and c can stand for Cartan generators, t_{diag}, or the ladder generators t_{ij}, t_{ji}. From the commutation relations for the ladder generators and the Cartan generators, it follows that the only non-zero values of $f^{a,b,c}$ exist when no more than one of the generators is diagonal. The specific non-zero cases are explored in more detail below.

It was demonstrated in [5, 6, 14] that at the two-loop level, including three and four vertex gluon interactions, and gluon-ghost interactions, all possible loop graphs generate contributions to the k-wall action, of the form

$$S_2 \sim \sum_{a,b,c} f^{a,b,c} f^{a,b,c} B_2(C_a) B_2(C_b). \tag{4.2}$$

Here B_2 is the second Bernoulli polynomial which is even in C_a

$$B_2(C_a) \sim \left(C_a^2 - |C_a| + \frac{1}{6}\right) \tag{4.3}$$

and the variables C_a are shorthand for the functions C_{ij} that encode all the q dependence

$$C_{ij} = A_{0\,i} - A_{0\,j} \sim q\left[(Y_k)_i - (Y_k)_j\right]. \tag{4.4}$$

It is obvious that $C_{ii} = 0$, while C_{ij} is only non-zero when i and j sit in different "sectors" of Y_k (recall from (2.11) that its elements live in two sectors taking only two possible values). Thus $C_{ij} = 0$ or $\pm q$ up to an overall factor.

With the above definitions and conventions, explicit computation of eq. (4.2) reveals Casimir-like scaling, like that at one-loop. This arises from summing all non-vanishing terms in (4.2) that have q dependence (i.e. ignoring all terms proportional to $B_2(0)^2$). The different non-trivial terms can be classified and accounted for as explicitly explained in the appendix. The final result of the combinatorics gives

$$S_2 \sim Nk(N-k)\left[B_2(q)^2 + 2B_2(q)B_2(0)\right]. \tag{4.5}$$

As explained in the appendix, the two key technical reasons for Casimir scaling in the final result are: one, that all q dependence in (4.2) arises from terms where at least one of the two indices a and b are off-diagonal generators. Secondly, and perhaps more importantly, the combination $B_2(C_a)B_2(C_b)$ is an even function of the C_a.

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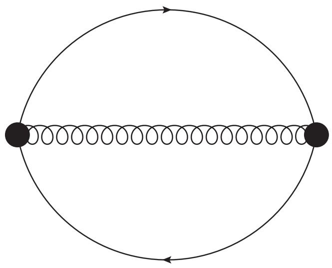

Figure 1: A two-loop fermion contribution to the effective action.

4.2 Argument for Casimir scaling in $\mathcal{N} = 4$ SYM at 2-loops

The two factors outlined in the previous section, coupled with the structure constants, effectively guarantee Casimir scaling in pure Yang-Mills. For this to be present in $\mathcal{N} = 4$ theory, the same factors must come into play. The propagators for the adjoint scalars in $\mathcal{N} = 4$ SYM, are equivalent (up to an overall factor) to the ghost propagators in pure YM. Therefore we expect the inclusion of the adjoint scalars, to not change the Casimir scaling at two loops.

For the adjoint fermions of the $\mathcal{N} = 4$ theory, a SUSY-based argument can be employed. With periodic boundary conditions, all perturbative fermionic and bosonic contributions to the effective potential for a constant (slowly varying) A_0 background will cancel due to supersymmetry. Since the bosonic fluctuations at two-loop yield Casimir scaling of the effective potential, the fermionic contributions will exhibit the same.

When the boundary conditions on the fermions are changed so that we have a thermal interpretation, the anti-periodic boundary conditions on fermions will only lead to a shift in q dependence, $q \to q' = q \pm 1/2$ due to the change in the Matsubara modes. This shift would leave all other overall scaling factors intact and we expect

$$S_2^F \sim (f^{ij,ji,\,\mathrm{diag}})^2 B_2(C_{ij}^F) B_2(C_{ji}) + \text{permutations} \tag{4.6}$$

where C_{ij}^F is the shifted difference,

$$C_{ij}^F \sim (A_{0\,i} - A_{0j}) \pm \frac{1}{2} \sim q \pm \frac{1}{2}. \tag{4.7}$$

Such a term would arise from the two-loop graph shown in figure 1.

This would then imply

$$\begin{aligned} S_2^F &\sim \frac{1}{2} B_2(q) \left[B_2\left(q + \frac{1}{2}\right) + B_2\left(q - \frac{1}{2}\right)\right] k(N-k) \\ &= \frac{1}{2} q^2 (1-q)^2 \left[\left(\frac{1}{2} + q\right)^2 \left(\frac{1}{2} - q\right)^2 + \left(-\frac{1}{2} + q\right)^2 \left(\frac{3}{2} - q\right)^2 \right] k(N-k) \end{aligned} \tag{4.8}$$

representing the pair of gluon-fermion vertices in figure 1. These arguments make it plausible that Casimir scaling of k-wall tensions persists at the two-loop level in $\mathcal{N} = 4$ SYM. However, there is no reason why this should continue to be the case beyond two-loops

5. $\mathbb{Z}_N$ domain walls at strong coupling

We now turn to the domain walls in strongly coupled $\mathcal{N} = 4$ theory in the large N limit, at finite temperature. The deconfined phase of the four dimensional field theory at strong coupling is described by Type IIB string theory in the Schwarzschild black hole in $AdS_5 \times S^5$ [15]. In the Euclidean picture, the conformal boundary of the geometry is $\mathbb{R}^3 \times S^1$, the boundary S^1 being identified with the Euclidean thermal circle of the strongly coupled field theory. The spontaneous breaking of the $\mathbb{Z}_N$ center symmetry in the deconfined phase follows from the fact that the thermal circle shrinks smoothly to zero size at a radial coordinate corresponding to the horizon of the Lorentzian black hole. The resulting black hole cigar can be wrapped by finite action string world-sheet instantons resulting in a non-zero Polyakov loop that spontaneously breaks the $\mathbb{Z}_N$ symmetry.

5.1 D-string as domain wall

With the $\mathbb{Z}_N$ center symmetry spontaneously broken it should be possible to identify the N distinct "vacua" of the theory in the IIB string dual. This question was raised and addressed in [12]. Shifting the phase of the Polyakov loop by the Nth roots of unity, moves us through the N possible ground states of the deconfined theory. In the string dual this is realized as a shift of the NS two-form field $B^{(2)}_{NS}$ integrated over the black hole cigar D_2,

$$\int_{D_2} B_{NS} \to \int_{D_2} B_{NS} + 2\pi k/N\,, \qquad k = 1, \ldots N. \tag{5.1}$$

Disc instantons wrapping the black hole cigar will then pick up a phase $\exp(i\,2\pi k/N)$, determining the VEV of the Polyakov loop.[3]

Across a domain wall, the phase of the Polyakov loop jumps. The argument of [12], shows that a D-string world-sheet $\Sigma \subset \mathbb{R}^3$ (and pointlike on the disc D_2 and the S^5) provides precisely such a jump. In particular, across Σ, the RR three form flux H_{RR} changes by one unit for a single D-string.

There is another argument establishing the connection between $\mathbb{Z}_N$ domain walls and the D-string. This exploits the direct relation between $\mathbb{Z}_N$ interfaces and spatial 't Hooft loops [10, 16]. The spatial 't Hooft loop operator $V(C)$, along a contour C, creates an infinitely thin tube of chromomagnetic flux along C. The spatial 't Hooft loop bounds a surface (a Dirac sheet) across which the gauge potential A_0 is discontinuous. The explicit perturbative computation of the expectation value of the 't Hooft loop in the deconfined phase can be shown to reduce to the domain wall calculations presented earlier in this paper. Specifically, the leading contribution to a large 't Hooft loop in $\mathbb{R}^3$ is proportional to the area A of the minimal surface bounded by C, $V(C) \sim \exp(-\mathcal{T}\,A)$. Thus the infinite volume domain wall tension $\mathcal{T}$ is computed by the spatial 't Hooft loop of infinite extent.

By the AdS/CFT correspondence, the 't Hooft loop is a Euclidean D-string world-sheet with disc topology, whose boundary traces the spatial loop in the field theory on

[3] The N vacua should be physically equivalent. This can be understood from the transformation of the term in the Type IIB low energy effective action $S = i\int B_{NS}\wedge\frac{H_{RR}}{2\pi}\wedge\frac{F_5}{2\pi}$. In the sector with $\int_{\mathbb{R}^3} H_{RR}/(2\pi) = 1$, the partition function will be invariant under (5.1), as $\delta S = i\frac{2\pi k}{N}\,N\int_{\mathbb{R}^3}\frac{H_{RR}}{2\pi} = 2\pi i k$, where we have used the fact that there are N units of five-form flux through the S^5.

the conformal boundary of the spacetime. In the AdS-Schwarzschild black hole geometry, the D-string world-sheet droops toward the interior of the spacetime. As the size of the loop is scaled up, most of the D-string worldsheet sits at the horizon where the bulk Euclidean geometry smoothly ends, and consequently the spatial 't Hooft loop exhibits an area law [17 − 19]. When the loop is taken to be of infinite extent, we obtain a Euclidean D-string world-sheet $\Sigma \subset \mathbb{R}^3$ located at the horizon of the Euclidean black hole. This is the $\mathbb{Z}_N$ interface.

5.2 k-wall tensions

The metric for the high temperature, Euclidean AdS-Schwarzschild black hole in $AdS_5 \times S^5$ is,

$$\begin{aligned} ds^2 =& R^2 \left[f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\, d\vec{x}^{\,2} + d\Omega_5^2 \right] \\ f(r) =& r^2 - \frac{\pi^4 T^4}{r^2}. \end{aligned} \tag{5.2}$$

where $R^4 = 4\pi(g_s N)\alpha'^2 = (g^2_{\rm YM} N)\alpha'^2$. The D1-brane world-sheet theory is described by the Dirac-Born-Infeld action in the absence of background $C^{(2)}_{RR}$ potential

$$S_{D1} = \frac{1}{2\pi\alpha'} \int d^2\sigma\, e^{-\Phi} \sqrt{\det{}^* g}. \tag{5.3}$$

The dilaton is constant with $e^{-\Phi} = 1/g_s = 1/g^2_{\rm YM}$. Assuming that the D-string worldsheet Σ is oriented along the $x - y$ plane, we choose the embedding $\sigma_1 = x$ and $\sigma_2 = y$. Then

$$\det{}^* g = R^4 r^4. \tag{5.4}$$

and to minimize the action, the D-string will sit at the smallest possible value of r, which, in this geometry is $r = \pi T$. We then find the tension of the $k = 1$ wall at strong coupling is

$$\mathcal{T}_1 = \frac{1}{2\pi\alpha' g_s} R^2 \pi^2 T^2 = 2\pi^2 \frac{N}{\sqrt{g^2_{\rm YM} N}} T^2. \tag{5.5}$$

Remarkably, the parametric dependence of this formula, on the 't Hooft coupling and N, closely resembles (3.41). The dependence on the temperature is guaranteed to be quadratic by the underlying conformal invariance of the $\mathcal{N} = 4$ theory. The N dependence is consistent with the domain wall being a D-brane in the large-N limit and the fact that the tension of the D-string in AdS is proportional to $1/\sqrt{g^2_{\rm YM} N}$ is also obvious from supergravity. What is interesting to note is that the formula at weak coupling also has the same dependence on the 't Hooft coupling.

For a collection of k D-strings, with $k \sim \mathcal{O}(1)$, the tension is simply k times that of a single D1-brane in the AdS-Schwarzschild background.

When the number of D-strings k, becomes of order N, in the large N limit, we can no longer think of the system as consisting of k separate D1-branes. In fact we expect the collection to blow up into a higher dimensional brane via an analogue of the dielectric effect [20] in the curved geometry. There are two possible blown up brane configurations to consider in the $AdS_5 \times S^5$ black hole geometry, carrying k units of D-string charge. At zero temperature, in $AdS_5 \times S^5$, electric Wilson loops in the k^{th} rank antisymmetric and symmetric tensor representations of SU(N), are computed by a D5-brane wrapped on an $S^4 \subset S^5$ and a D3-brane wrapping an $S^2 \subset AdS_5$, respectively [21–23, 25, 24]. Hence a collection of k D-strings, representing 't Hooft loops, could expand into wrapped NS5 and D3-branes, by S-duality.

5.2.1 The k-wall as a 5-brane

We expect that the correct configuration describing a k-wall is an expanded 5-brane. The 5-brane yields the k^{th} rank antisymmetric tensor representation of Wilson/'t Hooft loops; this is manifestly symmetric under $k \to N-k$, a property that we require from a candidate $\mathbb{Z}_N$ interface.[4] The intimate relationship between the baryon vertex (a 5-brane in the bulk) and flux tubes in the gauge theory (D- and F-strings in the bulk) [28–30] also naturally leads us to consider 5-branes as the candidates.

It is most convenient to first study an expanded probe D5-brane carrying k units of F1-string charge, and subsequently S-dualize to obtain the D-string domain wall. The action for the probe D5-brane has both Dirac-Born-Infeld and Wess-Zumino terms, and our analysis follows closely that in [22]

$$S = \frac{1}{(2\pi)^5 \alpha'^3 g_s}\left[\int dx\, dy d\Omega_4 \sqrt{\det({}^*g + 2\pi\alpha' F)} - i g_s \int 2\pi\alpha' F \wedge {}^*C_4\right] \tag{5.6a}$$

$$C^{(4)} = \frac{R^4}{g_s}\left[\frac{3}{2}(\alpha-\pi) - \sin^3\alpha\cos\alpha - \frac{3}{2}\cos\alpha\sin\alpha\right] \mathrm{Vol}(S^4). \tag{5.6b}$$

Here, $C^{(4)}$ is the relevant component of the RR four-form potential, proportional to the volume form on S^4. We have assumed that the D5-brane wraps an S^4 located at a polar angle $0 < \alpha \leq \pi$ inside the S^5. Thus the D5-brane has world-volume $\Sigma \times S^4$ where $\Sigma \subset \mathbb{R}^3$ is oriented in the $x-y$ plane. In addition, a world-volume electric field F_{xy} in the $x-y$ plane is switched on to endow the wrapped 5-brane with F-string charge. Since we are working in Euclidean signature, the electric field is imaginary, so it is useful to define $F_{xy} = iF$.

With this ansatz, using $R^4 = 4\pi(g_s N)\alpha'^2$, the D5-brane action is

$$S = \frac{N\sqrt{\lambda}}{3\pi^2}\int dx\, dy \left(\sin^4\alpha\sqrt{r^4 - \frac{4\pi^2 F^2}{\lambda}} - D(\alpha)\frac{2\pi F}{\sqrt{\lambda}}\right) \tag{5.7}$$

where we have defined the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$ and

$$D(\alpha) = -\frac{3}{2}(\alpha-\pi) + \sin^3\alpha\cos\alpha + \frac{3}{2}\cos\alpha\sin\alpha. \tag{5.8}$$

The equation of motion for the gauge field associated to F gives the total F-string charge k which is quantized; in particular the canonical momentum $\delta S/\delta F$ is the coupling of the

[4] 't Hooft/Wilson loops in the symmetric tensor representation do not have this symmetry property.

world-sheet to the B_{NS} field. Thus,

$$k = -\frac{\delta S}{\delta F} = \frac{2N}{3\pi}\left(\frac{2\pi F}{\sqrt{\lambda}}\frac{\sin^4\alpha}{\sqrt{r^4 - \frac{4\pi^2 F^2}{\lambda}}} + D\right). \tag{5.9}$$

Together with this, the equation of motion for the polar angle α determines the angle (and the size of the S^4) completely in terms of the string charge k,

$$\frac{2\pi F/\sqrt{\lambda}}{\sqrt{r^4 - \frac{4\pi^2 F^2}{\lambda}}} = -\cot\alpha \tag{5.10a}$$

$$\cos\alpha\sin\alpha - (\alpha - \pi) = \frac{k}{N}\pi. \tag{5.10b}$$

This equation implicitly fixes the location of the S^4 inside the S^5. Importantly, under the operation $k \to N - k$, the associated polar angle α is mapped to $\pi - \alpha$. All physical properties of the wrapped object are therefore invariant under $k \to N - k$, as necessary for the (magnetic) $\mathbb{Z}_N$ interface.

Since the world-volume electric field and the size of the internal S^4 is completely determined, it only remains to verify the radial coordinate of the F-string configuration. Plugging the solutions (5.10a), (5.10b), we have the effective action

$$S = \frac{N\sqrt{\lambda}}{3\pi^2}\int dx\, dy\, r^2\left[\sin^3\alpha + \frac{3}{2}\left(\frac{k}{N}\pi\right)\cos\alpha\right]. \tag{5.11}$$

5.2.2 Boundary terms

An extremely important point here is the inclusion of "boundary terms" in the problem at hand, an issue which was tackled in [22] and related references cited above. In these latter works, Wilson loops were being computed and the wrapped probe branes also had boundary terms in their action that were crucial and necessary. The interface under investigation here does not appear to have obvious boundary terms that need to be added since the entire world volume $\Sigma \subset \mathbb{R}^3$ does not actually extend to the boundary of AdS space. However, there is one type of boundary term that needs to be accounted for. This term acts as a Legendre transform, trading the world-volume gauge potential for its conjugate momentum and fixing the string charge k,

$$S_{\rm bdry} = k\int dx\, dy\, F. \tag{5.12}$$

so that the net action

$$S + S_{\rm bdry} = N\frac{\sqrt{\lambda}}{3\pi^2}\int dx\, dy\left(r^2\left[\sin^3\alpha + \frac{3}{2}\left(\frac{k}{N}\pi\right)\cos\alpha\right] + 3\pi^2\,\frac{k}{N}\frac{F}{\sqrt{\lambda}}\right). \tag{5.13}$$

The interpretation of the domain walls as infinitely large spatial Wilson/'t Hooft loops, makes it necessary to consider these boundary terms exactly as in [22, 23].

The inclusion of this term is also essential for guaranteeing invariance under $k \to N-k$. The equation of motion (5.10a) implies that $F = -r^2\sqrt{\lambda}\cos\alpha/2\pi$. Thus the complete

Lagrangian density in (5.13) only depends on r^2, and the action is minimized when $r = \pi T$. The resulting formula for the tension is the same as the action for 5-branes computing Wilson loops in the antisymmetric tensor representation,

$$T_{F1} = N\sqrt{\lambda}\frac{T^2}{3}\sin^3\alpha. \tag{5.14}$$

5.2.3 Tension of the 5-brane k-wall

Above, we deduced the tension of the wrapped D5-brane carrying k units of fundamental string charge. This object can be interpreted as a domain wall associated to the breaking of a magnetic $\mathbb{Z}_N$ symmetry of hot $\mathcal{N} = 4$ theory. S-duality on this yields the domain wall in the electric picture as a wrapped NS5-brane carrying k units of D-string charge.

Now we can S-dualize (5.14) by sending $g_s \to 1/g_s$, to obtain the tension of the k-domain wall at strong coupling, interpolating between two generic $\mathbb{Z}_N$ vacua in the high temperature $\mathcal{N} = 4$ theory:

$$\mathcal{T}_k = \frac{4}{3}\pi N^2 \frac{T^2}{\sqrt{g_{\rm YM}^2 N}}\sin^3\alpha\,, \qquad \cos\alpha\sin\alpha - (\alpha - \pi) = \frac{k}{N}\pi. \tag{5.15}$$

Obviously, this bears little resemblance to the weakly coupled theory (3.41). Nevertheless, there are a few significant remarks to be made. First, the dependence on the 't Hooft coupling is what one expects from a weakly curved string dual (SUGRA), and it is surprisingly in agreement with the weakly coupled Yang-Mills description (3.41). The second interesting feature of the tension at strong coupling is that when $k \sim \mathcal{O}(N)$, it scales as N^2, which is the scaling expected from a classical soliton in a large N theory (such as an NS5-brane in the IIB dual) — this feature also appears to be manifest at weak coupling from the Casimir scaling of the tension. Finally, for $k \ll N$, the strong coupling k-wall tension has an expansion in fractional powers of (k/N),

$$\mathcal{T}_k \simeq 2\pi^2 k\, N\,\frac{T^2}{\sqrt{\lambda}}\left(1 - C\left(\frac{k}{N}\right)^{2/3} + \ldots\right). \tag{5.16}$$

We round off our discussion of the strongly interacting theory with one additional aspect of the realization of domain walls as D-branes in the Type IIB theory. The $k = 1$ wall is a D-string worldsheet, pointlike on the transverse S^5. When $k \sim \mathcal{O}(N)$, the D-strings expand into an NS5-brane wrapping an $S^4 \subset S^5$. In both cases, the respective probe branes spontaneously break the SO(6) isometry of the S^5 to an SO(5) subgroup. While this feature is obvious from the dual string perspective, its interpretation in the gauge theory is somewhat obscure. In particular, the spontaneous breaking of the global symmetry suggests that at least classically there are massless fluctuations of the D-string worldsheet associated to fluctuations of a point, or an S^4, in the S^5. These classically massless internal zero modes are certainly not apparent in our weak-coupling $\mathbb{Z}_N$ instanton solutions, since the perturbative objects we discussed did not have any scalar profiles turned on.[5]

[5] In this context one should perhaps recall that if the effective domain wall theory were to be quantized, in the strongly coupled dual, the apparently massless Goldstone boson fluctuations should be generically rendered massive in the two dimensional world-volume theory of the domain wall.

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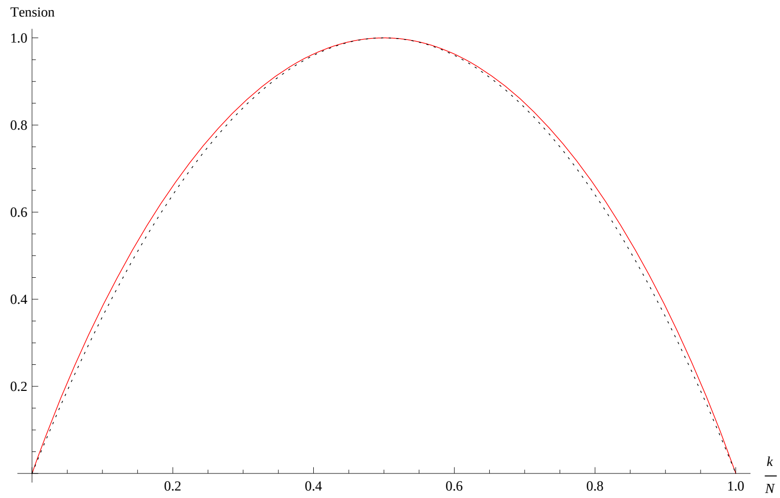

Figure 2: Casimir and $\sin^3$ behaviours. The solid red line is $\sin^3$ (strong coupling), whereas the dashed black line is the Casimir scaling (weak coupling).

The direct connection between the domain walls and 't Hooft loops may help in further clarifying this issue. In particular, we know that that the AdS/CFT correspondence does not provide us with direct access to Wilson/'t Hooft loops of the $\mathcal{N} = 4$ theory at strong coupling, but instead gives us the Wilson-Maldacena (or 't Hooft-Maldacena) loops [26, 27]

$$W(C) = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} e^{i \oint_C ds (A_\mu \dot{x}^\mu + \Phi_i \theta^i |\dot{x}|)}. \tag{5.17}$$

It would be interesting to compute the expectation values of these kinds of loops at weak coupling and high temperature. In particular, an interesting question is what kind of domain wall tension does the 't Hooft-Maldacena loop compute at high temperature, in the weakly coupled theory and how it differs from the standard $\mathbb{Z}_N$ domain wall at weak coupling. Specifically, are there classical solutions that have profiles for the scalars turned on, correlated in some way to the non-Abelian A_0 profile, providing the solutions with internal zero modes?

6. Conclusions

In this paper we discussed the tension of domain walls in the deconfined phase of $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^3 \times S^1$. While the tension is proportional to $k(N-k)$ at the one and two loop level at weak coupling, it exhibits a different behaviour at strong coupling. This is expected since already at three loops the $\mathbb{Z}_N$ domain wall tensions are not expected to exhibit a Casimir scaling. Nevertheless, a quantitative comparison of the weak and strong coupling behaviours of the tensions reveals intriguing features as we see below.

In figure 2 we have plotted the Casimir scaling (weak coupling behaviour) and the supergravity result (strong coupling) as a function of k/N for $N \to \infty$. The two graphs are normalized such that $\mathcal{T}_{k/N=1/2} = 1$. The maximum difference between the two graphs is about 4%.

Our results can be compared to lattice simulations, which were performed for the pure YM theory. Within the measurement error, the lattice results are compatible with a Casimir scaling [8, 9], even at low temperature (but above the deconfinement transition) where perturbation theory is not applicable and there is no reason to expect an exact Casimir scaling behaviour. It will be interesting to perform a more accurate simulation, in order to see a deviation from Casimir scaling at low temperatures. Although the results of this paper were obtained for $\mathcal{N} = 4$ super Yang-Mills, they might shed light on the expected tension of the pure Yang-Mills theory at strong coupling (low-temperatures). Qualitatively, we expect from our study that as the temperature of pure Yang-Mills theory decreases the ratio of the k-wall tension to the fundamental wall tension will increase, but by a very small amount (see figure 2).

We conclude this paper with several open problems. It would be a useful exercise to complete the two-loop calculation and to perform a three loops calculation of the k-wall tension in $\mathcal{N} = 4$ SYM. Another open question, relevant for both the pure and $\mathcal{N} = 4$ SYM, is to calculate the width of the domain wall in perturbation theory. We have also mentioned that the domain walls at strong coupling appear, at least classically, to have internal light degrees of freedom naturally associated to a spontaneous breaking of the SO(6) global symmetry. Their appearance is closely related to the Wilson-Maldacena loops in $\mathcal{N} = 4$ theory, which are natural in the strong coupling dual. It would be extremely interesting to firm up the connection between these loops and $\mathbb{Z}_N$ domain wall solutions at weak coupling in hot $\mathcal{N} = 4$ theory.

Finally, it will be interesting to find the field theory that lives on the $\mathbb{Z}_N$ domain walls. Since domain walls (at least at large N) are believed to be QCD D-branes [31], there should be a 1+1 dimensional field theory living on the domain walls, similarly to the Acharya-Vafa field theory that lives on the domain walls of $\mathcal{N} = 1$ SYM [32]. It would be fascinating to identify and show the existence of such a field theory, even in the case of pure YM theory. The answer to this question is certainly within reach for the $\mathcal{N} = 4$ theory at strong coupling.

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A. 2-loop combinatorics

Case I: $\boldsymbol{C_a, C_b \neq 0}$: C_a and C_b are only non-zero when a and b indices correspond to ladder generators. The remaining index on $f^{a,b,c}$ in (4.2) can either correspond to a Cartan (diagonal) generator or to a ladder operator. In the former case

$$f^{ij,ji,\mathrm{diag}} f^{ij,ji,\mathrm{diag}} B_2(C_{ij}) B_2(C_{ji}) \sim 2k(N-k) B_2(q)^2. \tag{A.1}$$

This result can be arrived at, by first noting that i, j must be in separate sectors of Y_k for C_{ij} and C_{ji} not to vanish. It is then clear that there are $2k(N-k)$ such terms which are identical since B_2 is an even function.

When the index c corresponds to a ladder generator, carefully following the combinatorics yields

$$f^{il,lj,ji}f^{il,lj,ji}B_2(C_{il})B_2(C_{lj}) \sim [k(N-k)(N-k-1)+k(k-1)(N-k)]\, B_2(q)^2. \quad \text{(A.2)}$$

Case II: $\boldsymbol{C_a = 0, C_b \neq 0}$ and $\boldsymbol{C_a \neq 0, C_b = 0}$: now either a *or* b can label a diagonal generator, or both a *and* b can correspond to off-diagonal generators. In the former case we have the contribution

$$f^{ij,\,\mathrm{diag},ji}f^{ij,\,\mathrm{diag},ji}B_2(C_{ij})B_2(C_{\mathrm{diag}}) \;\text{ or }\; f^{\mathrm{diag},ij,ji}f^{\mathrm{diag},ij,ji}B_2(C_{\mathrm{diag}})B_2(C_{ij}) \sim 4k(N-k)B_2(q)B_2(0). \quad \text{(A.3)}$$

Finally, when each of a, b and c map to off-diagonal, ladder generators

$$f^{il,lj,ji}f^{il,lj,ji}B_2(C_{il})B_2(C_{lj}) \quad \text{(A.4)}$$

Here i and j are in different sectors, therefore forcing l to be in a matching sector to one of them. Thus, either C_{il} or C_{lj} will vanish. Swapping sectors for i and j gives a factor of 2.

$$\rightarrow 2\,[k(N-k)(N-k-1)+k(k-1)(N-k)]\, B_2(q)B_2(0) \quad \text{(A.5)}$$

Total of I & II : summing all non-vanishing, q dependent terms from both cases yields:

$$\begin{aligned}&(2k(N-k)+[k(N-k)(N-k-1)+k(k-1)(N-k)])\,B_2(q)^2\\ &\quad +(4k(N-k)+2\,[k(N-k)(N-k-1)+k(k-1)(N-k)])\,B_2(q)B_2(0)\\ &\qquad = Nk(N-k)\left[B_2(q)^2+2B_2(q)B_2(0)\right]\end{aligned} \quad \text{(A.6)}$$

Casimir scaling remains at 2-loops in pure Yang-Mills.

There are two main factors above which lead to the Casimir-like scaling: Firstly, the only non-trivial q-dependence arises from either one, or both of C_a and C_b being non-zero. Secondly, and more importantly, the combined function $B_2(C_a)B_2(C_b)$ is even. Thus for the cases where the indices of f^2, a, b and c are off-diagonal (as in both of the cases above) the two contributions from l being in different sectors sum to give the scaling. Explicitly, consider a general function of C_a and C_b, $H(C_a, C_b)$, where all q-dependence vanishes only for $H(0, 0)$. Focusing on the analogous arguments to Case I:

$$f^{ij,ji,\mathrm{diag}}f^{ij,ji,\mathrm{diag}}H(C_{ij}, C_{ji}) = k(N-k)\,[H(q,-q)+H(-q,q)] \quad \text{(A.7)}$$

The diagonal contribution produces the Casimir scaling. However, for the off-diagonal contributions:

$$f^{il,lj,ji}f^{il,lj,ji}H(C_{il}, C_{lj}) = k(N-k)\,[(N-k-1)H(q,-q)+(k-1)H(-q,q)] \quad \text{(A.8)}$$

If $H(C_a, C_b)$ is not an even function, there is a departure from Casimir scaling. In particular, if $H(q,-q) = H(-q,q)$, then the two loop result is proportional to $k(N-k)$. The result, eq.(A.6), is the pure Yang-Mills result, however there is a plausibility argument that the Casimir scaling remains at 2-loops for $\mathcal{N}=4$ SYM.

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